

AN INVERSE ACKERMANNIAN LOWER BOUND ON THE LOCAL UNCONDITIONALITY CONSTANT OF THE JAMES SPACE

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ABSTRACT. The proof that the James space is not locally unconditional appears to be non-constructive, since it makes use of an ultraproduct construction. Using proof mining, we extract a constructive proof and obtain a lower bound on the growth of the local unconditionality constants.

1. INTRODUCTION

The failure of local unconditionality of the James space has been given [8] as an example of a theorem whose only known proof requires an ultraproduct argument (or, equivalently, nonstandard analysis). Recent developments in proof mining provide new insight into the relationship between standard and nonstandard proofs [20]. In particular, proof mining techniques can be used to “extract” a standard proof from a nonstandard one—that is, they provide a syntactic transformation which converts nonstandard proofs to standard ones.

In this paper we give a concrete example of these developments: a proof of the failure of local unconditionality of the James space which is explicit and constructive, and which provides the first (but very slow) lower bound on the local unconditionality constant of subspaces of the James space. This illustrates the following features of the modern understanding of the role of ultraproducts in proofs:

- Proofs of standard theorems which use ultraproducts can be systematically converted to explicit, constructive proofs which do not make use of ultraproducts.
- The main reason ultraproduct methods simplify proofs is that they allow the use of non-constructive theorems which have high *quantifier complexity*. In this example, the crucial step is a statement about the exchange of the order of a double limit.
- The functional interpretation provides a way to interpret statements about the ultraproduct as more complicated statements about the original space.

The usual proof of the local unconditionality of the James space has a measure theoretic flavor (in particular, it makes use of some of the theory of L_1 -space). To produce a constructive proof, we need a finitary version of these measure theoretic notions. The basic methods to do this were developed in the context of ergodic theory, most notably Tao's quantitative ergodic theory [16]. The most important idea is replacing convergence of limits with *metastable convergence* as introduced in [2, 17]. Formally, a sequence converges when

$$\forall \epsilon \exists n_\epsilon \forall M \geq n |a_{n_\epsilon} - a_M| < \epsilon.$$

The metastable version of this statement is

$$\forall \epsilon, F \exists n_{\epsilon, F} \forall m \in [n_{\epsilon, F}, F(n_{\epsilon, F})] |a_{n_{\epsilon, F}} - a_m| < \epsilon.$$

That is, given an accuracy ϵ and a function F , we can find an interval $[n, F(n)]$ on which the sequence is close to stable. The key point is that, in general, n_ϵ may not be computable from ϵ , but $n_{\epsilon, F}$ usually is computable from ϵ and F . Importantly, if we use the convergence of a sequence as an intermediate step in a proof, the actual bounds on our final theorem depend only on $n_{\epsilon, F}$ for a suitably chosen F .

The general relationship between statements and their constructive versions is given by the proof theoretic functional interpretation [1, 10]. In particular, the functional interpretation tells us how to convert more complicated statements into the corresponding constructive versions. The precise technique used to produce the results in this paper is an informal version of Kohlenbach's monotone functional interpretation [11]. In a companion paper, [19], we give an exposition of the motivating ideas in the context of a simpler example.

After introducing the definition of the James space and local unconditionality in Section 2, we outline the usual ultraproduct proof of the local unconditionality of the James space. In Section 3 we return to the issue of metastability, and describe the version of metastability we need for the particular convergence notions which turn up in the proof. In Section 4 we give the actual proof of local unconditionality.

2. THE JAMES SPACE

Definition 2.1. The *James space* J [9] consists of infinite sequences (α_n) of real numbers such that:

- (1) $\|(\alpha_n)\|_J = \frac{1}{\sqrt{2}} \sup [(\alpha_{p_1} - \alpha_{p_2})^2 + (\alpha_{p_2} - \alpha_{p_3})^2 + \cdots + (\alpha_{p_{m-1}} - \alpha_{p_m})^2 + (\alpha_{p_m} - \alpha_{p_1})^2]^{1/2}$ is bounded, where the supremum ranges over all m and all sequences $p_1 < p_2 < \cdots < p_m$, and
- (2) $\lim_n a_n = 0$.

The canonical basis for J consists of the vectors $\{e_i\}_{i \in \mathbb{N}}$ where e_i is the sequence $(0, \dots, 0, 1, 0, \dots)$ where the 1 occurs in the i -th position. We

write e_i^* for the corresponding dual functionals, so $e_j^*(e_i) = 1$ if $i = j$ and 0 otherwise.

Note that $\|e_i\|_J = \|e_i^*\|_{J^*} = 1$. For each i , we define $d_i = \sum_{j \leq i} e_j = (1, \dots, 1, 0, \dots)$. The sequence (d_i) provides a prototypical example of a sequence in J which is weakly Cauchy but not weakly convergent.

Definition 2.2. If (c_i) is a basis for a Banach space X , the *unconditional constant* for (c_i) is the supremum of

$$\frac{\|\sum_i \epsilon_i \alpha_i c_i\|_X}{\|\sum_i \alpha_i c_i\|_X}$$

where each $\epsilon_i \in \{-1, 1\}$. The unconditional constant for X , $ub(X)$, is the infimum of the unconditional constants of any basis of X .

The definitions above make sense for both finite and infinite dimensional X if we allow for the possibility that the unconditional constant is infinite in the infinite dimensional case.

Definition 2.3. A Banach space X has *local unconditional structure* if there is a constant B such that every finitely generated subspace of X has a basis with unconditional constant B .

Our main interest is the following theorem:

Theorem 2.4. *The James space does not have local unconditional structure.*

The proof (we follow the outline from [5]) comes from standard facts about the James space (as described in [5] or [14]) and two results, one which seems to have first appeared in [15], though it is due to Johnson and Tzafriri, and the second from [6]. (The proof given in the latter uses an argument based on the Hahn-Banach theorem rather than an ultraproduct construction, though the underlying idea is the same. A proof using ultraproducts explicitly is given, for instance, in [3].)

Those two results are quite general, so when specialized to the case of proving the James space does not have local unconditional structure, the proof simplifies:

Proof sketch. Suppose the James space had a local unconditional constant B . The ultrapower of a space with local unconditional constant is isomorphic to a Banach lattice, so we consider the ultrapower $J^{\mathcal{U}}$ as a Banach lattice. Consider the Banach lattice closure of the (d_i) ; call this X . X is separable and isomorphic (as a Banach lattice) to a subspace of $L_1(\Omega)$ for some measure space Ω . Let $\pi : X \rightarrow L_1(\Omega)$ be the corresponding injection and let $\pi^* : X^* \rightarrow L_1(\Omega)$ the corresponding dual; note that the range of π^* is contained in the dual of range of π . In particular, $y^*(x) = \int \pi^*(y^*)\pi(x)d\mu$.

The L_1 functions $\pi(d_n)$ converge weakly to some function f_∞ while the functions $\pi^*(e_p^*)$ converge weakly to some function g_∞ . Crucially, the products $\lim_n(\pi(d_n)\pi^*(e_p^*))$ and $\lim_p(\pi(d_n)\pi^*(e_p^*))$ also converge weakly, so the

limits exchange:

$$\begin{aligned} \lim_n \lim_p \int f_n g_p d\mu &= \lim_n \int f_n g_\infty d\mu = \int f_\infty g_\infty d\mu \\ &= \lim_p \int f_\infty g_p d\mu = \lim_p \lim_n \int f_n g_p d\mu. \end{aligned}$$

But we have

$$\int \pi^*(e_p^*) \pi(d_n) d\mu = \begin{cases} 1 & \text{if } p \leq n \\ 0 & \text{if } p > n \end{cases},$$

so $\lim_n \lim_p \int f_n g_p d\mu = 0$ while $\lim_p \lim_n \int f_n g_p d\mu = 1$, a contradiction. Therefore our assumption of local unconditionality was false. \square

This proof appears to be non-constructive—it tells us that for each constant B there is a sufficiently big finitely generated subspace X of J so that $ub(X) > B$, but it does not tell us what X is, or how big it must be. Perhaps surprisingly, the techniques in this proof are intrinsically constructive, but conceal the underlying quantitative information (with the benefit of substantially simplifying the proof). Below we make this quantitative information explicit.

3. PROOF MINING AND ULTRAPRODUCTS

We do not need the literal existence of the functions f_∞ and g_∞ to complete the proof; we really only need the fact that we can exchange the order of the limits in the double limit. Specifically, we need the following theorem:

Theorem 3.1. *Let $(f_n)_n$ and $(g_p)_p$ be sequences of L^1 functions such that*

- *the sequences $(f_n)_n$ and $(g_p)_p$ converge weakly,*
- *all the functions $f_n g_p$ are L_1 ,*
- *for each fixed n , the sequence $(f_n g_p)_p$ converges weakly, and*
- *for each fixed p , the sequence $(f_n g_p)_n$ converges weakly.*

Then

$$\lim_n \lim_p \int f_n g_p d\mu = \lim_p \lim_n \int f_n g_p d\mu.$$

For our actual application, we do not even need the existence of the limits. We need only:

Theorem 3.2. *Let $(f_n)_n$ and $(g_p)_p$ be sequences of L^1 functions such that*

- *the sequences $(f_n)_n$ and $(g_p)_p$ converge weakly,*
- *all the functions $f_n g_p$ are L_1 ,*
- *for each fixed n , the sequence $(f_n g_p)_p$ converges weakly, and*
- *for each fixed p , the sequence $(f_n g_p)_n$ converges weakly.*

Then for every n, p , and $\epsilon > 0$, there exist $m \geq n$ and $q \geq p$ such that for all $l \geq m$ and $r \geq q$, there exist $k \geq l$ and $s \geq r$ such that

$$\left| \int f_m g_s d\mu - \int f_l g_q d\mu \right| < \epsilon.$$

Note that this is a statement with high quantifier complexity: the conclusion has four blocks of alternating quantifiers—in logical notation, it is the following sentence

$$\forall n, p, \epsilon \exists m \geq n, q \geq p \forall l \geq m, r \geq q \exists k \geq l, s \geq r \left| \int f_m g_s d\mu - \int f_l g_q d\mu \right| < \epsilon,$$

which we will call σ .

In our context, we apply this theorem to an ultraproduct of the finite dimensional subspaces of the James space. Since this theorem is a true statement about the ultraproduct, there should be some fact about the finite dimensional subspaces of the James space corresponding to it. It turns out that this is precisely what the metastable version of the statement does for us. The metastable version of σ is:

For every $\epsilon > 0, p, n, \widehat{\mathbf{k}}, \widehat{\mathbf{r}}$ there are $m \geq n, q \geq p, \widehat{\mathbf{l}}, \widehat{\mathbf{s}}$ such that, if $\widehat{\mathbf{k}}(m, q, \widehat{\mathbf{l}}, \widehat{\mathbf{s}}) \geq m$ and $\widehat{\mathbf{r}}(m, q, \widehat{\mathbf{l}}, \widehat{\mathbf{s}}) \geq q$, then:

- $\widehat{\mathbf{s}}(\widehat{\mathbf{k}}(m, q, \widehat{\mathbf{l}}, \widehat{\mathbf{s}}), \widehat{\mathbf{r}}(m, q, \widehat{\mathbf{l}}, \widehat{\mathbf{s}})) \geq \widehat{\mathbf{r}}(m, q, \widehat{\mathbf{l}}, \widehat{\mathbf{s}}),$
- $\widehat{\mathbf{l}}(\widehat{\mathbf{k}}(m, q, \widehat{\mathbf{l}}, \widehat{\mathbf{s}}), \widehat{\mathbf{r}}(m, q, \widehat{\mathbf{l}}, \widehat{\mathbf{s}})) \geq \widehat{\mathbf{k}}(m, q, \widehat{\mathbf{l}}, \widehat{\mathbf{s}}),$
- $\left| \int f_m g_{\widehat{\mathbf{s}}(\widehat{\mathbf{k}}(m, q, \widehat{\mathbf{l}}, \widehat{\mathbf{s}}), \widehat{\mathbf{r}}(m, q, \widehat{\mathbf{l}}, \widehat{\mathbf{s}}))} d\mu - \int f_{\widehat{\mathbf{l}}(\widehat{\mathbf{k}}(m, q, \widehat{\mathbf{l}}, \widehat{\mathbf{s}}), \widehat{\mathbf{r}}(m, q, \widehat{\mathbf{l}}, \widehat{\mathbf{s}}))} g_q d\mu \right| < \epsilon.$

Notice that this statement has the form

$$\forall \vec{x} \exists \vec{y} \sigma^*(\vec{x}, \vec{y})$$

where $\vec{x} = \epsilon, p, n, \widehat{\mathbf{k}}, \widehat{\mathbf{r}}$ and $\vec{y} = m, q, \widehat{\mathbf{l}}, \widehat{\mathbf{s}}$.

Then the relationship we have is that σ is true in the ultraproduct exactly when

For every \vec{x} there is a \vec{Y} so that whenever J_K is a K -dimensional subspace of J with K sufficiently large, there is a $\vec{y} \leq \vec{Y}$ such that $\sigma^*(\vec{x}, \vec{y})$ is true in J_K .

In other words, the truth of σ in the ultraproduct is equivalent to the “uniform truth” of σ^* in the original structures. (We avoid the technical issue of what it means to have $\vec{y} \leq \vec{Y}$ in general, given that \vec{Y} involves functions [4, 12, 13], since we will only need a special case.)

σ^* is already rather complicated; fortunately, we only need the case where $n = p = 0$, $\widehat{\mathbf{k}}(m, q, \widehat{\mathbf{l}}, \widehat{\mathbf{s}}) = \max\{m, q + 1\}$ and $\widehat{\mathbf{r}}(m, q, \widehat{\mathbf{l}}, \widehat{\mathbf{s}}) = \max\{m + 1, q\}$. In this case the conclusion becomes

For every $\epsilon > 0$ there are $m, q, \widehat{\mathbf{l}}, \widehat{\mathbf{s}}$ such that:

- $\widehat{\mathbf{s}}(\max\{m, q + 1\}, \max\{m + 1, q\}) \geq \max\{m, q + 1\},$
- $\widehat{\mathbf{l}}(\max\{m, q + 1\}, \max\{m + 1, q\}) \geq \max\{m + 1, q\},$
- $\left| \int f_m g_{\widehat{\mathbf{s}}(\max\{m, q+1\}, \max\{m+1, q\})} d\mu - \int f_{\widehat{\mathbf{l}}(\max\{m, q+1\}, \max\{m+1, q\})} g_q d\mu \right| < \epsilon.$

Bounds on the sizes of the values $\widehat{\mathbf{s}}(\max\{m, q + 1\}, \max\{m + 1, q\})$ and $\widehat{\mathbf{l}}(\max\{m, q + 1\}, \max\{m + 1, q\})$ depend on the assumptions, however. For instance, we expect the size of $\widehat{\mathbf{s}}$ to depend on how rapidly the sequences

$(f_n)_n$ and $(g_p)_p$ converge. More precisely, we expect the size of $\widehat{\mathbf{s}}$ to depend on the rate of metastable convergence.

It will turn out that the sequences we need converge in a very strong way: they have bounded fluctuations.¹

Definition 3.3. The sequence (a_n) has *bounded fluctuations with bound* $f(\epsilon)$ if for every $\epsilon, \widehat{\mathbf{m}}, n$ there is an $m \in [n, \widehat{\mathbf{m}}^{f(\epsilon)}(n)]$ such that whenever $k, k' \in [m, \widehat{\mathbf{m}}(k)]$, $|a_k - a_{k'}| < \epsilon$.

This makes it possible to apply the main quantitative result from [18]. The result there is stated in terms of the *fast-growing hierarchy* of functions:

- $f_0(n) = n + 1$,
- $f_{m+1}(n) = f_m^n(n)$.

The exponent means that f_m is applied n consecutive times to n , so $f_1(n) = 2n$, $f_2(n) = 2^n n$, and so on. These functions grow very rapidly; in particular the function $f_\omega(m) = f_m(m)$ grows at roughly the same speed as the Ackermann function. This lets us state:

Theorem 3.4. *Suppose that:*

- *Each sequence $(f_n g_p)_p$ (for fixed n) and $(f_n g_p)_n$ (for fixed p) has bounded fluctuations with the uniform bound $8B^2(\lceil 1/\epsilon \rceil)^2$,*
- *For each n and any $\sigma \subseteq \Omega$ with $\mu(\sigma) < \epsilon/B2^n$, $\int_\sigma |f_n| d\mu < \epsilon$,*
- *For each p and any $\sigma \subseteq \Omega$ with $\mu(\sigma) < \epsilon/B2^p$, $\int_\sigma |g_p| d\mu < \epsilon$,*
- *For each n , $\|f_n\|_{L^1} \leq B$,*
- *For each p , $\|g_p\|_{L^1} \leq B$,*

Then for every E there exist $m < s$ and $q < l$ such that:

- $s, l \leq f_\omega(2^{22} B^4 \lceil 1/\epsilon \rceil^4 + 5)$, *and*
- $|(f_m g_s)(\Omega) - (f_l g_q)(\Omega)| < 20\epsilon$.

The theorem as stated in [18] has an additional technical condition regarding partitions into approximate level sets which is trivial in this case because f_n, g_p are explicitly presented as functions.

4. THE JAMES SPACE AS A FINITE MEASURE SPACE

4.1. Bounds on Fluctuations. Recall the sequence $d_i = \sum_{j \leq i} e_j = (1, \dots, 1, 0, \dots)$ in the James space where the first i elements of the sequence are 1 and the rest are 0. Since the sequence (d_i) is weakly Cauchy, we should be able to obtain bounds on the metastable weak convergence of this sequence. In this case we obtain an even stronger bound, which we will later convert this into a proof that the functions we are interested in have bounded fluctuations.

Lemma 4.1. *For any $\epsilon > 0$, any $k \geq 2\lceil 1/\epsilon \rceil^2$, any $n_0 < \dots < n_k$ and any y^* with $\|y^*\|_{J^*} \leq 1$, there is an $i < k$ so that $|y^*(d_{n_i}) - y^*(d_{n_{i+1}})| < \epsilon$.*

¹Saying that a sequence converges metastably is be equivalent to saying that a certain tree is well-founded. Having bounded fluctuations is the strengthening in which the tree specifically has height at most ω . [7] considers a similar issue.

Proof. Towards a contradiction, suppose not, and let $\epsilon, y^*, n_0 < \dots < n_k$ be a counterexample. We may divide the index set $I = [0, k)$ into two components, $I^> = \{i < k \mid y^*(d_{n_i}) > y^*(d_{n_{i+1}})\}$ and $I^< = [0, k) \setminus I^> = \{i < k \mid y^*(d_{n_i}) < y^*(d_{n_{i+1}})\}$. Clearly we have either $|I^>| > k/2$ or $|I^<| > k/2$; without loss of generality, we assume $|I^<| > k/2$ (the other case is symmetric).

Set $\hat{x} = \sum_{i \in I^>} (d_{n_{i+1}} - d_{n_i})$. So \hat{x} is the sum of those e_j such that $j \in \bigcup_{i \in I^>} (n_i, n_{i+1}]$. Therefore $\|\hat{x}\|_J = \frac{1}{\sqrt{2}} \sqrt{2|I^>|} = \sqrt{|I^>|}$ while

$$y^*(\hat{x}) = \sum_{i \in I^>} y^*(d_{n_{i+1}}) - y^*(d_{n_i}) \geq |I^>|\epsilon.$$

But this means $y^*(\hat{x}) \geq \sqrt{|I^>|}\epsilon \|\hat{x}\|_J > \sqrt{[1/\epsilon]^2}\epsilon \|\hat{x}\|_J \geq \|\hat{x}\|_J$, contradicting the fact that $\|y^*\| \leq 1$. \square

Analagously, consider the functional $e_\infty^* = \lim_{i \rightarrow \infty} e_i^*$; on J this is of course the functional which is constantly 0, but it becomes more useful on the ultrapower of J . The fact that e_∞^* is actually well-defined on the ultrapower of J is equivalent to the fact that the sequence $\lim_{i \rightarrow \infty} e_i^*$ converges metastably as in the following lemma:

Lemma 4.2. *For any $\epsilon > 0$, any $k \geq 2[1/\epsilon]^2$, any $p_0 < \dots < p_k$ and any x with $\|x\|_J \leq 1$, there is an $i < k$ so that $|e_{p_i}^*(x) - e_{p_{i+1}}^*(x)| < \epsilon$.*

Proof. Follows immediately from the definition of the norm $\|\cdot\|_J$: x is some sequence (x_n) . If $|e_{p_i}^*(x) - e_{p_{i+1}}^*(x)| > \epsilon$ for all $i < k$ then by definition

$$\begin{aligned} \|x\|_J &\geq \frac{1}{\sqrt{2}} \sqrt{\sum_{i < k} (x_{p_i} - x_{p_{i+1}})^2} \\ &= \frac{1}{\sqrt{2}} \sqrt{\sum_{i < k} (e_{p_i}^*(x) - e_{p_{i+1}}^*(x))^2} \\ &> \frac{1}{\sqrt{2}} \sqrt{k[1/\epsilon]^2} \\ &\geq 1. \end{aligned}$$

\square

4.2. A Finite Measure Space. Suppose the subspace J_K generated by $\{e_i\}_{i \leq K}$ has a basis $(\omega_i)_{i \leq K}$ with unconditional constant B . (Our use of the letter ω presages the fact that we will mostly be concerned with viewing the ω_i as elements in a measure space.) Let γ_i^* be the dual functionals corresponding to this basis, so any $x \in J_K$ satisfies $x = \sum_i \gamma_i^*(x) \omega_i$.

We can view $(\omega_i)_{i \leq K}$ as inducing a Banach lattice structure, with $x \leq y$ if for each $i \leq K$, $\gamma_i^*(x) \leq \gamma_i^*(y)$. We will not need this structure itself, but it motivates the following definitions.

For $x \in J_K$ we define $|x| = \sum_i |\gamma_i^*(x)| \omega_i$; since the (ω_i) are an unconditional basis, $\|x\|_J/B \leq \| |x| \|_J \leq B\|x\|_J$. Similarly, for $x^* \in J_K^*$ we define

$|x^*|(x) = \sum_i \gamma_i^*(x) |x^*(\omega_i)|$; we have $\|x^*\|_{J^*}/B \leq \|x^*\|_{J^*} \leq B\|x^*\|_{J^*}$. Note that if we choose $\epsilon_i \in \{-1, 1\}$ so that for each i , $\epsilon_i \gamma_i^*(x) x^*(\omega_i)$ is non-negative then we may set $x' = \sum_i \epsilon_i \gamma_i^*(x) \omega_i$ and we have

$$0 \leq |x^*|(|x|) = x^*(x') \leq \|x^*\|_{J^*} \|x'\|_J \leq B \|x^*\|_{J^*} \|x\|_J.$$

We fix canonical elements $d = \sum_{j \leq K} 2^{-j-1} |d_j|$ and $d^* = \sum_{j \leq K} 2^{-j-1} |e_j^*|$. Observe that

$$d^*(d) = \sum_{j, j' \leq K} 2^{-j-j'-2} |e_j^*|(|d_{j'}|) \leq \sum_{j, j' \leq K} 2^{-j-j'-2} B = B.$$

On the other hand,

$$d^*(d) \geq \sum_j 2^{-2j-2} |e_j^*(d_j)| \geq \frac{1}{4}.$$

We define a finite measure space, (Ω, μ) ; we take $\Omega = \{\omega_i\}_{i \leq K}$, and since Ω is atomic, it suffices to define $\mu(\{\omega_i\}) = \frac{\gamma_i^*(d)}{d^*(d)} d^*(\omega_i)$. We have

$$\mu(\Omega) = \frac{1}{d^*(d)} \sum_i \gamma_i^*(d) d^*(\omega_i) = \frac{d^*(d)}{d^*(d)} = 1.$$

We now define an embedding $\pi : J_K \rightarrow L_1(\Omega)$ by setting $\pi(x)$ to be the function

$$\sum_i \frac{d^*(d)}{\gamma_i^*(d)} \gamma_i^*(x) \chi_i$$

where χ_i is the characteristic function of the set $\{\omega_i\}$. That is, $\pi(x)$ is the function which, at the point ω_i , takes the value $\frac{d^*(d)}{\gamma_i^*(d)} \gamma_i^*(x)$. This definition has the convenient property that

$$\|\pi(x)\|_1 = \sum_i \frac{|\gamma_i^*(x)| d^*(d)}{\gamma_i^*(d)} \frac{\gamma_i^*(d) d^*(\omega_i)}{d^*(d)} = \sum_i |\gamma_i^*(x)| d^*(\omega_i) = d^*(|x|) \leq B \|x\|_J.$$

We also have an embedding $\pi^* : J_K^* \rightarrow L_1(\Omega)$ given by setting $\pi^*(x^*)$ to be the function

$$\sum_i \frac{d^*(d)}{d^*(\omega_i)} x^*(\omega_i) \chi_i.$$

Therefore

$$\begin{aligned} \int \pi^*(x^*) \pi(x) d\mu &= \sum_i \frac{x^*(\omega_i) d^*(d)}{d^*(\omega_i)} \frac{\gamma_i^*(x) d^*(d)}{\gamma_i^*(d)} \frac{\gamma_i^*(d) d^*(\omega_i)}{d^*(d)} \\ &= \sum_i \gamma_i^*(x) x^*(\omega_i) d^*(d) \\ &= x^*(x) d^*(d) \\ &\in [x^*(x)/4, B x^*(x)] \end{aligned}$$

and

$$\|\pi^*(x^*)\|_1 = \sum_i \frac{|x^*(\omega_i)|d^*(d)}{d^*(\omega_i)} \frac{\gamma_i^*(d)d^*(\omega_i)}{d^*(d)} = \sum_i |x^*(\omega_i)|\gamma_i^*(d) = |x^*|(d) \leq B\|x^*\|_{J^*}.$$

We define two sequences of $L^1(\Omega)$ functions: we set $f_n = \pi(d_n)$ and $g_p = \pi^*(e_p^*)$. Note that for any n we have $\|f_n\|_1 \leq B$ and for any p , $\|g_p\|_1 \leq B$.

We also have

$$\|f_n\|_\infty = \sup_i \left| \frac{\gamma_i^*(d_n)d^*(d)}{\gamma_i^*(d)} \right| \leq \sup_i \left| \frac{\gamma_i^*(d_n)d^*(d)}{2^{-n}\gamma_i^*(|d_n|)} \right| \leq B2^n$$

and

$$\|g_p\|_\infty = \sup_i \left| \frac{e_p^*(\omega_i)d^*(d)}{d^*(\omega_i)} \right| \leq \sup_i \left| \frac{e_p^*(\omega_i)d^*(d)}{2^{-p}|e_p^*(\omega_i)|} \right| \leq B2^p.$$

4.3. Quantitative Convergence and Continuity.

Lemma 4.3. *For any fixed p , the sequence $(\rho_n \lambda_p)_n$ has bounded fluctuations with bound $8B^2 \lceil 1/\epsilon \rceil^2$.*

Proof. Let $\epsilon > 0$, $\widehat{\mathbf{m}}, n, \sigma$ be given. Without loss of generality, assume $\widehat{\mathbf{m}}(n) \leq \widehat{\mathbf{m}}(n+1)$ for all n (if this is not the case, we replace $\widehat{\mathbf{m}}$ with $\widehat{\mathbf{m}}'(n) = \max_{n' \leq n} \widehat{\mathbf{m}}(n')$).

Consider the function $y^*(x) = \frac{1}{B} \int_\sigma \pi(x) \lambda_p d\mu$ and define a sequence inductively by $m_0 = n$ and if there is any $m \in [m_i, \widehat{\mathbf{m}}(m_i)]$ such that $|y^*(d_{m_i}) - y^*(d_m)| \geq \epsilon/2B$ then m_{i+1} is the least such m , otherwise $m_{i+1} = \widehat{\mathbf{m}}(m_i)$. By the monotonicity of $\widehat{\mathbf{m}}$, $m_{2B^2E^2} \leq \widehat{\mathbf{m}}^{2B^2E^2}(n)$. By Lemma 4.1 there is an $i < 8B^2 \lceil 1/\epsilon \rceil^2$ so that $|y^*(d_{m_i}) - y^*(d_{m_{i+1}})| < \epsilon/2B$. By the choice of m_{i+1} , it must be that $m_{i+1} = \widehat{\mathbf{m}}(m_i)$ and for every $k \in [m_i, \widehat{\mathbf{m}}(m_i)]$, $|y^*(d_{m_i}) - y^*(d_k)| \geq \epsilon/2B$.

Suppose there were some $k, k' \in [m_i, \widehat{\mathbf{m}}(m_i)]$ so that $|y^*(d_k) - y^*(d_{k'})| \geq \epsilon/B$. Then either $|y^*(d_{m_i}) - y^*(d_k)| \geq \epsilon/2B$ or $|y^*(d_{m_i}) - y^*(d_{k'})| \geq \epsilon/2B$. But this is a contradiction. Therefore for every $k, k' \in [m_i, \widehat{\mathbf{m}}(m_i)]$,

$$|\rho_{m_i}(\sigma) - \rho_m(\sigma)| = B|y^*(d_k) - y^*(d_{k'})| < \epsilon.$$

□

Similarly, using Lemma 4.2, we obtain

Lemma 4.4. *For any fixed n , the sequence $(\rho_n \lambda_p)_p$ has bounded fluctuations with bound $8B^2 \lceil 1/\epsilon \rceil^2$.*

Lemma 4.5. *For each n and any $\sigma \subseteq \Omega$ with $\mu(\sigma) < \epsilon/B2^n$, $\int_\sigma |\rho_n| d\mu < \epsilon$.*

Proof. Immediate since $\|\rho_n\|_\infty \leq B2^n$. □

Lemma 4.6. *For each p and any $\sigma \subseteq \Omega$ with $\mu(\sigma) < \epsilon/B2^p$, $\int_\sigma |\lambda_p| d\mu < \epsilon$.*

4.4. Putting it Together.

Theorem 4.7. *If $K \geq f_\omega(2^{29}B^4+5)$ then the local unconditionality constant of a basis of J_K is $> B$.*

Proof. Suppose J_K has a basis $(\omega_i)_{i \leq K}$ with unconditional constant $\leq B$. We construct f_n, g_p as described above, and apply Theorem 3.4 with $\epsilon = 1/80$ to obtain $m < s \leq K$ and $q < l \leq K$ so that

$$\left| \int (f_m g_s) d\mu - \int (f_l g_q) d\mu \right| < 1/4.$$

But since $m < s$, $\int (f_m g_s) d\mu = \int \pi(d_m) \pi^*(e_s^*) d\mu = e_s^*(d_m) d^*(d) = 0$. On the other hand, since $q < l$, $\int (f_l g_q) d\mu = e_q^*(d_l) d^*(d) \geq 1/4$, which is a contradiction. \square

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